4.4 Indeterminate Forms and L' Hospitâl's Rule

In this section we will discuss how to take the limit of functions that previously seemed to be impossible to compute.

If we have a limit of the form: $\mathbf{lim}_{x\to a}\frac{f(x)}{g(x)}$ where both **f(x) → 0** and **g(x) → 0** as **x → a**, then this limit may or may not exist and is called an **indeterminate form** of type $\frac{\mathbf{0}}{\mathbf{0}}$.

In addition, if we have a limit of the form: $\mathbf{lim}_{x\to a}\frac{f(x)}{g(x)}$ where both **f(x) →** $\pm\infty$ and **g(x) →** $\pm\infty$, then the limit may or may not exist and called an **indeterminate form** of type $\frac{\infty}{\infty}$.

To help us with these problems, we introduce **L' Hospitâl's Rule** (pronounced **lō-pē-tall**)

L' Hospitâl's Rule: Suppose ***f*** and ***g*** are differentiable and ***g'(x) ≠ 0*** on an open interval ***I*** that contains ***a*** (except possibly at ***a***). Suppose that

$$\mathbf{lim}_{x\to a} f(x) = \mathbf{0} \text{ and } \mathbf{lim}_{x\to a} g(x) = \mathbf{0} \text{ or}$$

$$\mathbf{lim}_{x\to a} f(x) = \pm\infty \text{ and } \mathbf{lim}_{x\to a} g(x) = \pm\infty$$

(In other words we have an indeterminate for of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$) Then,

$$\mathbf{lim}_{x\to a}\frac{f(x)}{g(x)} = \mathbf{lim}_{x\to a}\frac{f'(x)}{g'(x)}$$

(provided the limit on the right side of the equation exists or is $\pm\infty$.)

Note 1: L' Hospitâl's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided the given conditions are satisfied.

Note 2: L' Hospitâl's Rule is also valid for one-sided limits and for limits at infinity for negative infinity; that is $\mathbf{x\to a^+}$, $\mathbf{x\to a^-}$, $\mathbf{x\to \infty}$, or $\mathbf{x\to -\infty}$.

Example: Using L' Hospitâl's Rule: Evaluate the following limit.

$\lim_{x\to 0}\frac{\sqrt{9+3x}-3}{x}$ Substituting **x =0** into this function produces the indeterminate form $\frac{0}{0}$.

Let $f(x) = \sqrt{9+3x} - 3$ and $g(x) = x$

then $f'(x) = \frac{1}{2}(9-3x)^{-\frac{1}{2}}\cdot 3 = \frac{3}{2\sqrt{9-3x}}$ and $g'(x) = 1$

Applying **L'Hospitâl's Rule**, we have: (The notation for applying **L'Hospitâl's Rule** is [**H**])

$$\lim_{x\to 0}\frac{\sqrt{9+3x}-3}{x} = \lim_{x\to 0}\frac{\frac{3}{2\sqrt{9-3x}}}{1} = \frac{\mathbf{1}}{\mathbf{2}}.$$

L'Hospitâl's Rule requires evaluating the $\mathbf{lim}_{x\to a}\frac{f'(x)}{g'(x)}$. It may be that this second limit is another indeterminate form to which **L'Hospitâl's Rule** may be applied again.

Example: Evaluate the following limit:

$\lim_{x\to 0}\frac{e^x-x-1}{x^2}$ by substitution we get $\frac{e^0-0-1}{0^2}=\frac{0}{0}$ [H] $\to \lim_{x\to 0}\frac{e^x-1}{2x}$ by substitution we get $\frac{e^0-1}{2(0)}=\frac{0}{0}$ [H] $\to$ $\lim_{x\to 0}\frac{e^x}{2}$ by substitution we get $\frac{e^0}{2}=\frac{1}{2}$ $\therefore$ $\mathbf{\lim_{x\to 0}\frac{e^x-x-1}{x^2}=\frac{1}{2}}$.

Example: Evaluate the following limit:

$\lim_{x\to\frac{\pi}{2}^-}\frac{1+\tan x}{\sec x}$ by substitution we get $\frac{1+\infty}{\infty}=\frac{\infty}{\infty}$ [H] $\to \frac{sec^2 x}{\sec(x)\tan(x)}(simplify)=\frac{\sec(x)}{\tan(x)}=\frac{\frac{1}{\cos(x)}}{\frac{\sin(x)}{\cos(x)}}=\frac{1}{\sin(x)}$

$\lim_{x\to\frac{\pi}{2}^-}\frac{1}{\sin(x)}=1$ $\therefore$ $\mathbf{\lim_{x\to\frac{\pi}{2}^-}\frac{1+\tan x}{\sec x}=1}$

Before using **L'Hospitâl's Rule** make sure to check for indeterminate cases. **L'Hospitâl's Rule** will not work unless we get the indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate Products:

If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty\ (or\ -\infty)$, then it isn't clear what the value of $\lim_{x\to a}[f(x)\cdot g(x)]$, if any, will be. There is a struggle between $\boldsymbol{f}$ and $\boldsymbol{g}$ as to which one's limit will "win" or take over. If $\boldsymbol{f}$ wins then the limit will be 0, if $\boldsymbol{g}$ wins then the limit will be ∞ (or -∞). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of** $0\cdot\infty$. We can deal with it by writing the product as a quotient: $f\cdot g=\frac{f}{\frac{1}{g}}\ or\ \frac{g}{\frac{1}{f}}$. This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which is necessary to use **L'Hospitâl's Rule.**

Example: Evaluate $\lim_{x\to\infty}x^2\sin\left(\frac{1}{4x^2}\right)$ $\lim_{x\to\infty}x^2\sin\left(\frac{1}{4x^2}\right)$ by substitution we get the indeterminate product form of $\infty\cdot 0$. If we rewrite this by dividing by the reciprocal of x^2 we will get the indeterminate form of $\frac{0}{0}$. $\lim_{x\to\infty}\frac{sin\left(\frac{1}{4x^2}\right)}{\frac{1}{x^2}}=\frac{0}{0}$ [H] $\to \lim_{x\to\infty}\frac{cos\left(\frac{1}{4x^2}\right)\cdot\frac{1}{4}(-2x^{-3})}{-2x^{-3}}$ (simplify) $=\lim_{x\to\infty}\frac{1}{4}\cdot\cos\left(\frac{1}{4x^2}\right)=$ $\frac{1}{4}\lim_{x\to\infty}cos\left(\frac{1}{4x^2}\right)$ as $x\to\infty$ $\frac{1}{4x^2}\to 0=\frac{1}{4}\cos(0)=\frac{1}{4}\cdot 1=\mathbf{\frac{1}{4}}$ $\therefore$ $\mathbf{\lim_{x\to\infty}x^2\sin\left(\frac{1}{4x^2}\right)=\frac{1}{4}}$

Indeterminate Differences:

If $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$, then the limit $\lim_{x\to a}[f(x)-g(x)]$ is in the indeterminate form of type $\infty-\infty$. Again we have to manipulate the problem into a quotient, (by using a common denominator, or rationalizing, or factoring out a common factor, etc…) so that we have the indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example: Evaluate $\lim_{x\to\infty} x - \sqrt{x^2 - 3x}$ We can see that as **x** approaches infinity, both terms, x *and* $\sqrt{x^2 - 3x}$ approach infinity. Therefore, this problem has the indeterminate difference form of $\infty - \infty$. Using lots of algebra manipulation we can rewrite. We can factor an $\boldsymbol{x^2}$ out inside the square root.

$$\lim_{x\to\infty} x - \sqrt{x^2 - 3x} = \lim_{x\to\infty}\left(x - \sqrt{x^2\left(1 - \frac{3}{x}\right)}\right) = \lim_{x\to\infty}\left(x - x\sqrt{1 - \frac{3}{x}}\right) = \lim_{x\to\infty} x\left(1 - \sqrt{1 - \frac{3}{x}}\right) =$$

$\lim_{x\to\infty}\left(\frac{1-\sqrt{1-\frac{3}{x}}}{\frac{1}{x}}\right) = \frac{0}{0}$ **[H]** → Let $\frac{1}{x} = t$ and replace limit as **x** →∞ with the limit as $\boldsymbol{t \to 0^+}$

$\lim_{t\to 0^+}\frac{(1-\sqrt{1-3t})}{t}$ **[H]** → $\lim_{t\to 0^+}\frac{\frac{3}{2\sqrt{1-3t}}}{1} = \frac{3}{2}$ ∴ $\boldsymbol{\lim_{x\to\infty} x - \sqrt{x^2 - 3x} = \frac{3}{2}}$

Example: Evaluate $\boldsymbol{\lim_{x\to 0^+}\cot(x) - \frac{1}{x}}$ As $x \to 0^+$, both $\cot(x)$ and $\frac{1}{x}$ approach ∞, therefore this problem has the indeterminate form $\infty - \infty$. To rewrite this problem write $\cot(x)$ as $\frac{\cos(x)}{\sin(x)}$

So $\lim_{x\to 0^+}\cot(x) - \frac{1}{x} = \lim_{x\to 0^+}\frac{\cos(x)}{\sin(x)} - \frac{1}{x}$ (find a common denominator . . . $x \cdot \sin(x)$)

$$= \lim_{x\to 0^+}\frac{x\cos(x) - \sin(x)}{x\sin(x)} = \frac{0\cdot\cos(0) - \sin(0)}{0\cdot\sin(0)} = \frac{0}{0} \quad \textbf{[H]}$$

$$= \lim_{x\to 0^+}\frac{-x\sin(x) + \cos(x) - \cos(x)}{x\cos(x) + \sin(x)} = \lim_{x\to 0^+}\frac{-x\sin(x)}{x\cos(x) + \sin(x)} = \frac{0}{0} \quad \textbf{[H]}$$

$$= \lim_{x\to 0^+}\frac{-x\cos(x) - \sin(x)}{-x\sin(x) + \cos(x) + \cos(x)} = \lim_{x\to 0^+}\frac{-x\cos(x) - \sin(x)}{-x\sin(x) + 2\cos(x)} = \frac{0}{2} = \mathbf{0}$$

Therefore, $\boldsymbol{\lim_{x\to 0^+}\cot(x) - \frac{1}{x} = 0}$

Indeterminate Powers:

Several indeterminate forms arise from the limit $\lim_{x\to a}[f(x)]^{g(x)}$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ indeterminate type 0^0
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ indeterminate type ∞^0
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ indeterminate type 1^∞

Each of these cases can be solved by one of two ways: 1.) taking the natural logarithm:

If $y = [f(x)]^{g(x)}$, then $\boldsymbol{\ln(y) = g(x)\cdot\ln[f(x)]}$ or by writing the function as an exponential:

$\boldsymbol{[f(x)]^{g(x)} = e^{g(x)\cdot\ln[f(x)]}}$

Example: Evaluate $\mathbf{\lim_{x\to 0^+} x^x}$ This limit has the indeterminate form of type 0^0. We can rewrite this as:

$x^x = e^{x\cdot\ln(x)}$ but $x \cdot \ln(x)$ has the form of $0 \cdot \infty$, however; we can rewrite $x \cdot \ln(x)$ as $\frac{\ln(x)}{\frac{1}{x}} = \frac{-\infty}{-\infty}$. Now that we have that expression in an indeterminate form we can use L'Hospitâl's Rule **[H]**. (Note that using limit laws we can write: $\lim_{x\to 0^+} e^{x\cdot\ln(x)} = e^{\lim_{x\to 0^+} x\cdot\ln(x)}$). So now we take the limit of the exponent using **[H]**.

$\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0$ So now we have that

$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\cdot\ln(x)} = e^0 = 1$